

## 2.1 – Determinants by Cofactor Expansion

Definition 1: If  $A$  is a square matrix, then the **minor of entry  $a_{ij}$**  is denoted by  $M_{ij}$  and is defined to be the determinant of the submatrix that remains after the  $i$ th row and  $j$ th column are deleted from  $A$ . The number  $(-1)^{i+j}M_{ij}$  is denoted by  $C_{ij}$  and is called the **cofactor of entry  $a_{ij}$** .

Ex: Find all the minors and cofactors of  $A = \begin{bmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{bmatrix}$ .

Definition 2: If  $A$  is an  $n \times n$  matrix, then the number obtained by multiplying the entries in any row or column of  $A$  by the corresponding cofactors and adding the resulting products is called the **determinant of  $A$** , and the sums themselves are called **cofactor expansions of  $A$** . That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(cofactor expansion along the  $j$ th column)

$$= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(cofactor expansion along the  $i$ th row)

Ex: Compute the determinant of the matrix  $A$  above.

11. Use the arrow technique of Figure 2.1.1 to evaluate the determinant.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$$

**Theorem 2.1.1** If  $A$  is an  $n \times n$  matrix, then regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

16. Find all values of  $\lambda$  for which  $\det(A) = 0$ .

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

**Theorem 2.1.2** If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal), then  $\det(A)$  is the product of the entries on the main diagonal of the matrix; that is,  
 $\det(A) = a_{11}a_{22} \cdots a_{nn}$ .

31. Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$